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Vacuum energy in quantum field theory with external potentials concentrated on planes

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Abstract. In the presence of an idealized potential on two parallel planes represented by two one-dimensional δ-functions at $x_3=-d/2$ and $x_3=+d/2$ we discuss the Feynmann propagators for relativistic scalar and spinor fields. These propagators take into account bound states, scattering states and resonances. The Casimir energy for this configuration is calculated. For massive fields the Casimir force decreases exponentially with rising distances. In the scalar case we find an attractive force and in the spinor case a repulsive force. An attempt to treat the same problem for a massive scalar field using non-relativistic quantum field theory leads to a vanishing Casimir force.

1. Introduction

The vacuum energy of quantized fields is an interesting and fundamental quantity. If there are external parameters in the theory, then the vacuum energy depends on these parameters, which can lead to observable consequences. The most popular example is the Casimir effect [1] where the vacuum energy depends on the distance between two conducting plates. The corresponding force has been observed [2]. To date, many configurations and boundary conditions have been considered (see the reviews [3, 4] for example).

In the usual treatment of the Casimir effect one investigates the electromagnetic field only and considers the plates as perfect conductors represented by the corresponding boundary conditions. It is, however, possible to idealize a metallic plate with respect to the behaviour of the electrons by a potential well (Sommerfeld's potential pot model), especially by an idealized potential well represented by a δ-function. This makes sense because the energy states in such a potential contain only one bound state (to each degree of freedom) and the usual continuum states [5, 6]. So far estimating further contributions to the Casimir force we are able to represent the two metallic plates with respect to the electron field by two external δ-type potential wells separated by the distance d [7].

In this paper we calculate the Casimir energy in the presence of the potential

$$eA_0(x)=a(\delta(x_3-d_1)+\delta(x_3-d_2)) \tag{1.1}$$

for massive and massless scalar and spinor fields. Because of the difficulties of the formulation of a scalar field theory with a singular external potential the scalar model

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has very restricted validity. The action of potential (1.1) is reformulated as boundary conditions to the fields at the positions of the plates. In this connection the resulting new boundary conditions can be viewed as a generalization of the usually taken Dirichlet boundary conditions controlled by a new parameter a. However, if we think about idealized metallic plates then we expect two contributions. First we have to take into account the vacuum energy and second the energy of the filled levels. Here we will study the vacuum energy only. First calculations of this problem have been performed by Mamaev and Trunov [7] for the special case of relativistic scalar fields under the influence of repulsive δ-potentials.

Here we use a consequent field-theoretic formulation for the calculation of the Casimir effect. We discuss the known expressions [8] of the propagators in the presence of the δ-potential. These propagators take into account bound states, scattering states and resonances. They are represented by closed expressions in momentum space. As is well known, this is possible for a very limited number of potentials (homogeneous fields, periodic fields, Coulomb field, special potential wells) [9] only. More solvable examples are known in non-relativistic field theory (see [10] and references therein).

The vacuum energy is calculated as the vacuum expectation value of the energy momentum tensor. Using point splitting as the regularization procedure we can directly insert the derived Green functions. As a result we obtain that for massive fields the Casimir force decreases exponentially with increasing distance. This justifies in principle the standard procedure for the determination of the Casimir force by taking into account the photon field only. All other possible fields (electron field, proton field, etc) lead to exponentially decreasing contributions which can be neglected for large distances. In the scalar case we find an attractive force and in the spinor case a repulsive force. This may be in accordance with speculations on supersymmetry [11].

An attempt to treat the same problem for a massive scalar field using non-relativistic quantum field theory leads to a vanishing Casimir force. Usually one considers that the Casimir effect is determined by the infrared part of the zero point energy. For massive particles this part of the spectrum should be well approximated by a non-relativistic treatment. From our result it may follow that also in the massive case the change in the infrared spectrum from relativistic theory cannot be approximated sufficiently well by non-relativistic theory.

The paper is organized as follows. We introduce the necessary field-theoretic notation in section 2. The propagators are discussed, taking into account the solutions of the field equation in the next section. The Casimir energy is calculated in the fourth section. The corresponding investigations in non-relativistic quantum field theory under the influence of the potential $A_0(x)$ (see (1.1)) are given in section 5.

2. Basic notation

In this section we give the necessary field-theoretical notation, which is close to that used in standard textbooks [12]. In the scalar case the starting point is the Lagrange density

$$\mathscr{L}(x)=\partial_\mu\varphi^*(x)\partial^\mu\varphi(x)+\left(m^2-\sum_i 2a_i(x_3-d_i)\right)\varphi^*(x)\varphi(x). \tag{2.1}$$

If we compare it with a Lagrangian with minimal coupling to a gauge field $A_0\sim\delta(x_3-d)$ then we have taken into account the most singular part of this interaction,

$\varphi^*\varphi\delta(x_3)\delta(x_3) \sim \text{const}\ \varphi^*\varphi\delta(x_3)$, in a regularized way. The parameter a, which can be understood as the strength of the potential, has the dimension of an inverse length. This is the simplest procedure to obtain a well-defined problem for concentrated potentials within scalar field theory [7, 8]. Note that in this way the charge sensitivity of the external potential is lost and the potential couples in the same manner to particles and antiparticles. This is one reason why the scalar model has very limited validity, but it allows a simple treatment in comparison with the following spinor model. The necessary formulae are: the field equation

$$\left(\partial^2 - 2\sum_i a\delta(x_3-d_i) + m^2\right)\varphi(x) = 0$$

the causal propagator

$$\left(\partial^2 - 2\sum_i a\delta(x_3-d_i) + m^2\right){}^s D^c(x,y) = \delta(x-y) \tag{2.2}$$

$$\langle 0|\mathbf{T}\varphi(x)\varphi^*(y)|0\rangle = -\mathrm{i}^s D^c(x,y) \tag{2.3}$$

and the energy-momentum tensor

$$T_{\mu\nu}(x) = \partial_\mu\varphi(x)\partial_\nu\varphi^*(x) + \partial_\nu\varphi(x)\partial_\mu\varphi^*(x) - g_{\mu\nu}\mathscr{L}(x). \tag{2.4}$$

The formulae for the spinor field are similar. As the Lagrange density we choose

$$\mathscr{L}(x) = \overline{\psi(x)}(\mathrm{i}\hat{\Delta} - m)\psi(x)$$

($\hat{\Delta} \equiv \Delta_\mu\gamma^\mu$, $\Delta_\mu = \partial\mu - \mathrm{i}eA\mu$), then the propagator $S^c(x,y)$ obeys

$$(\mathrm{i}\hat{\Delta} - m)^s S^c(x,y) = \delta(x-y)$$

as well as

$$\langle 0|\mathbf{T}\overline{\psi(x)}\psi(y)|0\rangle = \mathrm{i}S^c(x,y) \tag{2.5}$$

whereas the energy-momentum tensor takes the form

$$T_{\mu\nu}(x) = \frac{\mathrm{i}}{2}\overline{\psi(x)}\gamma_\mu\left(\frac{\vec{\partial}}{\partial x_\nu} - \frac{\overleftarrow{\partial}}{\partial x_\nu}\right)\psi(x). \tag{2.6}$$

We consider the potential $A_\mu(x)$, given by (1.1), which is concentrated on parallel planes perpendicular to the x_3 axis, intersecting it at $x_3 = d_i$ $(i = 1, 2)$. Without loss of generality we choose $d_{1,2} = \pm d/2$, so that d is the distance between the planes. Because the potential is concentrated on the planes it is possible to rewrite it as a boundary condition. This means that the equation of motion is considered for $x_3 \neq d_i$ only and accompanied by boundary conditions at $x_3 = d_i$. The form of the energy-momentum tensor remains unchanged because it has to be considered for $x_3 \neq d_i$.

In the scalar case the boundary condition is

$$\frac{\partial}{\partial x_3}\varphi(x)\Big|_{x_3=d_i+0} - \frac{\partial}{\partial x_3}\varphi(x)\Big|_{x_3=d_i-0} + 2a\varphi(x)\Big|_{x_3=d_i} = 0 \tag{2.7}$$

and the field is assumed to be continuous at $x_3 = d_i$.

In the spinor case the field equation is

$$\left(\mathrm{i}\hat{\partial} - m + \sum_{i=1}^{2} a\delta(x_3-d_i)\gamma^0\right)\psi(x) = 0.$$

Again it can be replaced by the free field equation supplemented by a boundary condition [8, 13],

$$\boldsymbol{R}\psi(x)|_{x_3=d_i-0}=\psi(x)|_{x_3=d_i+0} \tag{2.8}$$

without any problems. $\boldsymbol{R}$ has the properties of a rotation matrix:

$$\boldsymbol{R}=\exp(\mathrm{i}\gamma^0\gamma^3\vartheta)$$

where $\tan(\vartheta/2)=a/2$. Note that the parameter a, i.e. the strength of the potential, is dimensionless in this case.

3. Solutions and propagators for δ-potentials

Here we investigate the propagators, taking into account the solutions of the field equations. We start with the scalar case and discuss the more complicated spinor case afterwards. All the considered solutions factorize into a plane wave part in the (x_0, x_1, x_2) directions and other x_3-dependent functions.

3.1. Scalar fields

In the case with one δ-potential located at $x_3=0$ there exists a set of symmetric scattering states

$$\varphi_k^{\mathrm{sy}}(x_3)=\sqrt{\frac{2}{\pi}}\cos\left(k|x_3|+\tan^{-1}\left(\frac{a}{k}\right)\right) \tag{3.1}$$

$(k=0,\ldots,\infty)$, a set of antisymmetric scattering states

$$\varphi_k^{\mathrm{as}}(x_3)=\sqrt{\frac{2}{\pi}}\sin(kx_3) \tag{3.2}$$

$(k_{(\mathrm{as})}=0,\ldots,\infty)$, and for $0<a<m$ a bound state solution $(k=-\mathrm{i}a)$

$$\varphi_{\mathrm{bound}}(x_3)=\sqrt{a}\exp(-a|x_3|). \tag{3.3}$$

Note that these solutions appear for particles and antiparticles in the same way. For $a<0$ there is no bound state solution because the potential is repulsive. For $a>m$ and if $k_1^2+k_2^2<a^2-m^2$ the corresponding energy $k_0=\sqrt{k_1^2+k_2^2-a^2+m^2}$ becomes imaginary. In this case the binding energy of the bound state is larger than the energy gap between particle and antiparticle states. This situation is called 'level diving' and is well known (see [9] for example). We do not consider this situation here and assume $a<m$.

In the case of two δ-potentials there are similar solutions. The symmetric scattering states are

$$\varphi_k^{\mathrm{sy}}(x_3)=\frac{1}{\sqrt{\pi}}\begin{cases}\dfrac{\cos(kd/2+\delta_{(\mathrm{sy})})}{\cos(kd/2)}\cos(kx_3) & \text{for } |x_3|<\dfrac{d}{2}\\[2ex] \cos(k|x_3|+\delta_{(\mathrm{sy})}) & \text{for } |x_3|>\dfrac{d}{2}\end{cases} \tag{3.4}$$

$(k=0,\ldots,\infty)$, where δ is given by

$$\delta_{(\mathrm{sy})}=-k\frac{d}{2}+\tan^{-1}\left[\tan\left(k\frac{d}{2}\right)-\frac{2a}{k}\right].$$

The antisymmetric scattering states are

$$\varphi_k^{\mathrm{as}}(x_3)=\frac{1}{\sqrt{\pi}}\begin{cases}\dfrac{\sin(kd/2+\delta_{(\mathrm{as})})}{\sin(kd/2)}\sin(kx_3) & \text{for } |x_3|<\dfrac{d}{2}\\ \sin(k|x_3|+\varepsilon(x_3)\delta_{(\mathrm{as})}) & \text{for } |x_3|>\dfrac{d}{2}\end{cases}\tag{3.5}$$

with $\delta_{(\mathrm{as})}=-kd/2+\cot^{-1}[\cot(kd/2)-2a/k]$ [3].

For $a>0$ there are, additionally, two bound state solutions. The symmetric bound state

$$\varphi_{\mathrm{bound}}^{\mathrm{sy}}(x_3)=\frac{1}{\sqrt{(2/a)\,\mathrm{e}^{\gamma_{\mathrm{sy}}d}+2d}}\begin{cases}\mathrm{e}^{\gamma_{\mathrm{sy}}x_3}+\mathrm{e}^{-\gamma_{\mathrm{sy}}x_3} & \text{for } |x_3|<\dfrac{d}{2}\\ (1+\mathrm{e}^{\gamma_{\mathrm{sy}}d})\,\mathrm{e}^{-\gamma_{\mathrm{sy}}|x_3|} & \text{for } |x_3|>\dfrac{d}{2}\end{cases}\tag{3.6}$$

with $\gamma_{\mathrm{sy}}=a(1+\mathrm{e}^{-\gamma_{\mathrm{sy}}d})$, and the antisymmetric bound state

$$\varphi_{\mathrm{bound}}^{\mathrm{as}}(x_3)=\frac{1}{\sqrt{(2/a)\,\mathrm{e}^{\gamma_{\mathrm{as}}d}-2d}}\begin{cases}\mathrm{e}^{\gamma_{\mathrm{as}}x_3}-\mathrm{e}^{-\gamma_{\mathrm{as}}x_3} & \text{for } |x_3|<\dfrac{d}{2}\\ (-1+\mathrm{e}^{\gamma_{\mathrm{as}}d})\varepsilon(x_3)\,\mathrm{e}^{-\gamma_{\mathrm{as}}|x_3|} & \text{for } |x_3|>\dfrac{d}{2}\end{cases}\tag{3.7}$$

with $\gamma_{\mathrm{as}}=a(1-\mathrm{e}^{-\gamma_{\mathrm{as}}d})$.

Next, we list the expressions for the propagators [8]. We denote the propagator with boundary conditions by ${}^{\mathrm{s}}D^{\mathrm{c}}(x,y)$ and split it into the free space part and the boundary-dependent part according to

$${}^{\mathrm{s}}D^{\mathrm{c}}(x,y)=D^{\mathrm{c}}(x,y)-\overline{D^{\mathrm{c}}}(x,y).\tag{3.8}$$

We note from (2.2) that $\overline{D^{\mathrm{c}}}(x,y)$ obeys the homogeneous equation

$$(\Box+m^2)\overline{D^{\mathrm{c}}}(x,y)=0\tag{3.9}$$

everywhere except on the boundary, where it is determined by the boundary condition. Because of the symmetry properties of the system it is useful to apply a mixed representation. To derive it we start with the free field propagator $D^{\mathrm{c}}(x,y)$ and carry out the k_3 integration [14], and get

$$D^{\mathrm{c}}(x,y)=\int\frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\,\frac{\mathrm{i}}{2\Gamma}\,\mathrm{e}^{\mathrm{i}k_\alpha(x^\alpha-y^\alpha)+\mathrm{i}\Gamma|x_3-y_3|}\tag{3.10}$$

where in the following $\alpha=0,1,2$ and

$$\Gamma=\sqrt{k_0^2-k_1^2-k_2^2+m^2+\mathrm{i}\varepsilon}.\tag{3.11}$$

In the presence of one δ-potential we get the representation

$$\overline{D^{\mathrm{c}}}(x,y)=\int\frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}k_\alpha(x^\alpha-y^\alpha)}\,\frac{\mathrm{i}}{2\Gamma}\,\frac{a}{a+\mathrm{i}\Gamma}\,\mathrm{e}^{\mathrm{i}\Gamma(|x_3|+|y_3|)}.\tag{3.12}$$

Let us study this representation in some detail. First we remark that, in the limiting case $a\to 0$, $\overline{D^{\mathrm{c}}}(x,y)$ vanishes and ${}^{\mathrm{s}}D^{\mathrm{c}}(x,y)$ (3.8) reduces to the free field propagator, as it must. Next, we consider the integrand as a function of k_0 for fixed k_1 and k_2. The integration path γ lies in the complex k_0-plane. There are two cuts starting from $k_0=\pm\sqrt{k_1^2+k_2^2+m^2}$ arising from the square root in Γ (see (3.11)). The upper sheet is defined by $\operatorname{Im}\Gamma>0$ for real k_0 between the cuts (i.e. for $|k_0|\leqslant\sqrt{k_1^2+k_2^2+m^2}$). The integration path γ lies just in this sheet. The cuts represent the scattering states (3.1) and (3.2). Now, from the δ-potential two poles arise at $k_0=\pm\sqrt{k_1^2+k_2^2+m^2-a^2}$ as solutions of the equation $\Gamma=\mathrm{i}a$. For $0<a<m$ these poles lie on the upper sheet and correspond to stable bound states. For $a>m$ (and $k_1^2+k_2^2<a^2-m^2$) they move to the imaginary axis and level diving occur. For $a<0$ they are still present, but on the lower sheet ($\operatorname{Im}\Gamma=a<0$), and represent resonances. The limit $|a|\to\infty$ can be performed just for negative a where the poles cannot conflict with the integation path γ. In this case $D^c(x,y)$ (see (3.12)) turns to the known expression satisfying the Dirichlet boundary [14].

In the presence of two δ-potentials the boundary-dependent part of the propagator takes the representation [8]

$$\overline{D^c}(x,y)=\frac{a}{2}\int\frac{\mathrm{d}k_\alpha}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}k_\alpha(x^\alpha-y^\alpha)}\times\frac{(1-\mathrm{i}a/\Gamma)\,\mathrm{e}^{-\mathrm{i}\Gamma(|x_3-d_1|+|y_3-d_1|)}+(\mathrm{i}a/\Gamma)\,\mathrm{e}^{-\mathrm{i}\Gamma(|x_3-d_1|+|y_3-d_2|+d)}+(d_1\leftrightarrow d_2)}{(\Gamma-\mathrm{i}a)^2+a^2\,\mathrm{e}^{2\mathrm{i}\Gamma d}}. \tag{3.13}$$

The validity of this representation can be checked. We now consider the following properties. First, for $d_1=d_2$ (i.e. $d=0$) we reproduce (3.12) for one δ-potential of strength $2a$. Further, let us consider the analytic structure in the k_0-plane. Again there exist two cuts corresponding to the scattering states. From the zeros of the denominator on the RHS of (3.13), i.e. from the solutions of the equation

$$\mathrm{e}^{\mathrm{i}\Gamma d}=\pm(1+\mathrm{i}\Gamma/a) \tag{3.14}$$

we obtain poles where the sign + (respectively −) corresponds to the antisymmetric (respectively symmetric) bound states. For pure imaginary $\Gamma=\mathrm{i}y/d$ (y real) these equations are

$$\mathrm{e}^{-y}=\pm(1-y/ad). \tag{3.15}$$

For $ad>-a^*$ with $a^*=0.278\,464$ (a^* is a solution of $a^*=1+\ln a^*$) they have real solutions $y^\pm(ad)$, shown in figure 1.

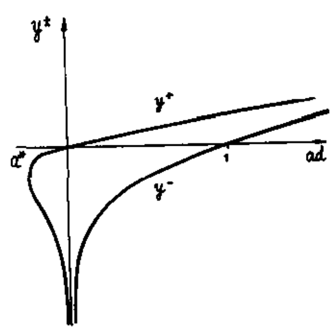

Figure 1. The solutions $y^\pm(ad)$ of (3.14).

These solutions have the following interpretation for different values of the parameter ad. By means of $k_0 = \pm\sqrt{k_1^2 + k_2^2 + m^2 - y_\pm^2}$ they determine the location of the two poles for each solution in the k_0 plane. For $0 \leq y_\pm < \sqrt{k_1^2 + k_2^2 + m^2}$ the poles lie in the upper sheet on the real axis between the cuts, for $y_\pm > \sqrt{k_1^2 + k_2^2 + m^2}$ the corresponding poles move to the imaginary k_0 axis and level diving occurs. For $y_\pm < 0$ the poles move to the real axis on the lower sheet and become unphysical. It is interesting to note the possibility of $y_+ > 0$ and $y_- < 0$ (for the same value of ad) in the region $0 < ad < 1$. In this case there is one bound state and one resonance for each particle and antiparticle only.

To get the general solutions of (3.14) we represent Γ by

$$\Gamma = (x + \mathrm{i}y)/d \tag{3.16}$$

with real x and y. Then, (3.14) splits into two equations:

$$\begin{aligned} \mathrm{e}^{-y} \cos x &= \pm(1 - y/ad) \\ \mathrm{e}^{-y} \sin x &= \pm x/ad. \end{aligned} \tag{3.17}$$

Substituting y from the second equation into the first equation we get

$$x \cot x + \ln|\sin x/x| = ad - \ln ad \tag{3.18}$$

which has a set of solutions $x_n \sim \pi n$ $(n = 1, 2, \ldots)$. The corresponding y_n are all negative. Thus the corresponding poles in the k_0 plane lie on the lower sheet. They represent the resonances occurring between the planes with the δ-potentials. For $ad \to \infty$ they move to the real k_0 axis with $x_n = \pi n$, $y_n = 0$. This is the case for Dirichlet boundary conditions and (3.13) turns out to be the corresponding expression [14]. In this light we can regard the boundary conditions corresponding to δ-functions as a possible modification of the Dirichlet boundary condition controlled by the parameter a. In this physical picture conditions (2.7) and (2.8) could model the penetrability of boundaries.

3.2. Spinor fields

The spinor case can be treated in close analogy to the scalar one. However, the formulae are more complicated. We first give the propagator [8] which is needed for the calculation of the Casimir effect and then we will consider the bound state solutions. In the free field case the propagator $S^c(x, y)$ reads in the preferred mixed representation as

$$S^c(x, y) = \int \frac{\mathrm{d}^3 p_\alpha}{(2\pi)^3} \frac{\mathrm{e}^{-\mathrm{i}p_\alpha(x^\alpha - y^\alpha)}}{2\mathrm{i}\Gamma} [\check{p} + m - \gamma^3 \Gamma \varepsilon(x_3 - y_3)]\, \mathrm{e}^{\mathrm{i}\Gamma|x_3 - y_3|}. \tag{3.19}$$

Note that we use here and in the following $\hat{p} \equiv p_\mu \gamma^\mu$ $(\mu = 0, 1, 2, 3)$ and $\check{p} \equiv p_\alpha \gamma^\alpha$ $(\alpha = 0, 1, 2)$. For the general case with the boundary conditions (2.8) we make the ansatz

$${}^{s}S^c(x, y) = S^c(x, y) - \overline{S^c}(x, y) \tag{3.20}$$

with

$$\begin{aligned} \overline{S^c}(x, y) = \int \frac{\mathrm{d}^3 p_\alpha}{(2\pi)^3} \frac{\mathrm{e}^{-\mathrm{i}p_\alpha(x^\alpha - y^\alpha)}}{2\mathrm{i}\Gamma} \sum_{i,j=1}^{2} [\check{p} + m - \gamma^3 \Gamma \varepsilon(x_3 - d_i)]\, \mathrm{e}^{\mathrm{i}\Gamma|x_3 - d_i|} \\ \times K_{ij} [\check{p} + m - \gamma^3 \Gamma \varepsilon(d_j - y_3)]\, \mathrm{e}^{\mathrm{i}\Gamma|d_j - y_3|} \end{aligned} \tag{3.21}$$

where i, j are the planes with δ-potential and K_{ij} is a spinor-valued matrix to be determined from the boundary conditions as

$$K_{ij}=\frac{\alpha}{2m}A^{-1}\delta_{ij}+\frac{\alpha^2\delta}{2mN}A^{-1}\begin{pmatrix}\alpha\delta P_+A^{-1}P_-; -P_+\\ -P_-; \alpha\delta P_-A^{-1}P_+\end{pmatrix}_{ij}A^{-1}. \tag{3.22}$$

The notation is $P_\pm=(m+\check{p}\pm\gamma^3\Gamma)/2m$, $\alpha=-(2im/\Gamma)\tan(\vartheta/2)$, $\delta=e^{i\Gamma d}$

$$A=\gamma^0+\frac{\alpha}{2}(P_++P_-)\qquad A^{-1}=\frac{1}{\lambda}\left(\gamma^0-\frac{\alpha}{2}(2-P_+-P_-)\right) \tag{3.23}$$

with

$$\lambda=1-\tan\left(\frac{\vartheta}{2}\right)^2-\frac{2ip_0}{\Gamma}\tan\left(\frac{\vartheta}{2}\right). \tag{3.24}$$

The $P_\pm$ are projection operators and fulfil the relations

$$\begin{gathered}P_\pm^2=P_\pm\qquad P_\pm P_\mp=\frac{\check{p}}{m}P_\mp\qquad P_\pm\gamma^0P_\pm=\frac{p_0}{m}P_\pm\\ P_\mp A^{-1}P_\pm A^{-1}P_\mp=\frac{p_1^2+p_2^2+m^2}{\lambda^2m^2}P_\mp\qquad P_\pm A^{-1}P_\mp=\frac{1}{\lambda}P_\pm\gamma^0P_\mp.\end{gathered} \tag{3.25}$$

The quantity N in the denominator is defined as

$$N=1-\left(\frac{\alpha\delta}{\lambda m}\right)^2(p_1^2+p_2^2+m^2).$$

Its zeros determine the spectrum, like (3.14) in the scalar case. The condition $N=0$ can be rewritten in the form

$$p_0+i\Gamma\cot(\vartheta)=\pm\sqrt{p_1^2+p_2^2+m^2}\,e^{i\Gamma d}. \tag{3.26}$$

We add the spinor propagator in the case of one δ-potential (at $x_3=d_1$, for example). It is given by (3.20) and (3.21) with

$$K_{11}=\frac{\alpha}{2m}A^{-1}\qquad K_{ij}=0 \text{ for } i \text{ or } j\neq 1. \tag{3.27}$$

The spectrum follows from the condition $\lambda=0$, which can be rewritten as

$$p_0=\varepsilon(\vartheta)\sqrt{p_1^2+p_2^2+m^2}\cos(\vartheta). \tag{3.28}$$

Here, in contrast to the scalar case, the symmetry between the particle and antiparticle is broken. Therefore, for a potential attractive with respect to the particles, the pole with $p_0>0$ lies in the upper sheet of the p_0 plane and corresponds to a bound state, whereas the pole with $p_0<0$ lies in the lower sheet and corresponds to a resonance state for the antiparticle. An essential difference from the scalar case is that here no level diving occurs because $|p_0|<m$ for all p_1, p_2 and for all values of the strength $a=2\tan(\vartheta/2)$ of the δ-potential. Similar considerations are true for (3.26) representing the case of two δ-potentials.

For completeness we derive here the bound state solutions $\psi_{(n)}(x_3)\,e^{-ipx}/2\pi$. This is a non-trivial task. The simplest way is to extract these solutions from the known representation of the propagator. For this reason we remark that the propagator ${}^sS^c(x,y)$ can be represented in the form

$${}^sS^c(x,y)=\int\frac{d^3p_\alpha}{(2\pi)^3}\,e^{-ip_\alpha(x^\alpha-y^\alpha)}\sum_{(n)}\frac{\psi_{(n)}(x_3)\overline{\psi_{(n)}(y_3)}}{p_0-(E_{(n)}-E_{(n)})(1+i\varepsilon)} \tag{3.29}$$

($\varepsilon > 0$). Here (n) denotes the quantum number which characterizes the solutions in addition to p_1 and p_2, and $E_{(n)}$ are the corresponding energy values. The solutions $\psi_{(n)}(x_3)$ obey the equation

$$\left(\check{p} - m + \mathrm{i}\gamma^3 \frac{\partial}{\partial x^3}\right)\psi_{(n)}(x_3) = 0 \qquad \text{for } x_3 \neq d_i.$$

In the case of one δ-potential the index (n) for the bound states consists of two spin states distinguished by $\sigma = \pm 1$. The energy is given by $p_0 = \varepsilon(\vartheta)\sqrt{p_1^2 + p_2^2 + m^2}\cos(\vartheta)$ (see (3.28)). In the case of two δ-potentials it includes two solutions of (3.26) (denoted by $E^{\pm}$) for the energy, each of which has two spin states (with $\sigma = \pm 1$). Furthermore, the bound states are characterized by $|p_0| < m$ and, consequently, $\Gamma = \sqrt{p_0^2 - p_1^2 - p_2^2 - m^2} \equiv \mathrm{i}\gamma$ with real γ. Now we compare the above-derived representations (3.20), (3.21) of ${}^{s}S^{c}(x, y)$ with (3.29). Closing the integration path for p_0 in (3.29) as well as in (3.21) we obtain a contribution from the poles at $p_0 = E^{\pm}$ which contains the bound states. Of course, there are further contributions related to the scattering states. From the residuum of the poles we get

$$\sum_{\substack{(n)\\ \text{bound}}} \psi_{(n)}(x_3)\overline{\psi_{(n)}(x_3)}$$

$$= \frac{1}{2\gamma} \operatorname{Res} \sum_{i,j=1}^{2} \left(\check{p} + m + \mathrm{i}\gamma^3 \frac{\partial}{\partial x^3}\right) \mathrm{e}^{-\gamma|x_3 - d_i|} K_{ij} \left(\check{p} + m - \mathrm{i}\gamma^3 \frac{\partial}{\partial y^3}\right) \mathrm{e}^{-\gamma|y_3 - d_j|}$$

where the residuum is taken at $p_0 = E^{\pm}$. It is clear that it is just $\overline{S^{c}}(x, y)$ which contributes to the pole term. In the case of one δ-potential we have $(n) = \sigma = \pm 1$ and make the ansatz

$$\psi_\sigma(x_3) = \left(\check{p} + m + \mathrm{i}\gamma^3 \frac{\partial}{\partial x^3}\right) \mathrm{e}^{-\gamma|x_3|}\psi_\sigma(p_1, p_2).$$

Taking into account (3.27) we get

$$\sum_{\sigma = \pm 1} \psi_\sigma(p_1, p_2)\overline{\psi_\sigma(p_1, p_2)} = \frac{\alpha}{4\gamma m \partial\lambda/\partial p_0}\left(\gamma^0 + \frac{\alpha}{2}\frac{\check{p} - m}{m}\right).$$

Now it is easy to prove that

$$\psi_\sigma(p_1, p_2) = \frac{1}{\mathcal{N}} \begin{pmatrix} (p_{0_\sigma} + m)\cos(\vartheta)\chi_\sigma \\ -(\sigma_1 p_1 + \sigma_2 p_2)\sin(\vartheta)\chi_\sigma \end{pmatrix} \tag{3.30}$$

with

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathcal{N} = \frac{2m}{\alpha}\sqrt{2\gamma\lambda'(p_0 + m)\cos(\vartheta)}.$$

In the case of two δ-potentials a similar ansatz reads as

$$\psi_\sigma(x_3) = \sum_{i=1}^{2} \left(\check{p} + m + \mathrm{i}\gamma^3 \frac{\partial}{\partial x_3}\right) \mathrm{e}^{-\gamma|x_3 - d_i|}\psi_{i,\sigma} \tag{3.31}$$

with $p_0 = E^{\pm}$. For the spinors $\psi_{i,\sigma}$ we get from K_{ij} (see (3.22))

$$\psi_{i,\sigma} = A^{-1}\frac{1}{\mathcal{N}} \begin{pmatrix} 1 \\ -(\alpha\delta/\lambda)P_{-}\gamma^0 \end{pmatrix}_i \begin{pmatrix} (p_0 + m)\chi_\sigma \\ -(\sigma_1 p_1 + \sigma_2 p_2 + \mathrm{i}\gamma\sigma_3)\chi_\sigma \end{pmatrix} \tag{3.32}$$

with

$$\chi_\sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sigma(p_2 - \mathrm{i}p_1)/\sqrt{p_1^2 + p_2^2} \end{pmatrix}$$

where $\mathcal{N}$ is a normalization factor.

4. Calculation of the Casimir energy

The aim is the calculation of the distance-dependent part of the vacuum energy. For simplicity we first consider scalar field theory. One possibility is the investigation of the energy–momentum tensor $T_{\mu\nu}$ (see (2.4)). Its vacuum expectation value can be expressed by the T-product of the fields. Using (2.2) and the field equation we get

$$\langle 0|T_{00}|0\rangle = 2\frac{\partial}{\partial x_0}\frac{\partial}{\partial y_0}\frac{1}{\mathrm{i}}(D^{\mathrm{c}}(x,y) - \overline{D^{\mathrm{c}}}(x,y))\big|_{x=y} \tag{4.1}$$

where $x \neq y$ is used as a regularization. The first term on the RHS of (4.1) contains the free field contribution; because of its distance independence we omit it. In further calculations we take into account $\overline{D^{\mathrm{c}}}$ only. But this contribution is not finite. As distance-independent divergent contributions we can furthermore eliminate terms connected with the δ-potentials of two individual planes.

First we consider the contribution from one δ-potential. Inserting (3.12) for $\overline{D^{\mathrm{c}}}(x,y)$ into (4.1) we get

$$\langle 0|T_{00}^{1\delta}|0\rangle = -\int \frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\frac{k_0^2}{\Gamma}\frac{a}{a+\mathrm{i}\Gamma}\,\mathrm{e}^{2\mathrm{i}\Gamma|x_3|}.$$

and after a Wick rotation $k_0 \to \mathrm{i}k_4$

$$\langle 0|T_{00}^{1\delta}|0\rangle = -\int \frac{\mathrm{d}_E^3k_\beta}{(2\pi)^3}\frac{k_4^2 a}{\gamma(a-\gamma)}\,\mathrm{e}^{-2\gamma|x_3|}$$

with $\gamma = \sqrt{k_4^2 + k_1^2 + k_2^2 + m^2}$, $\beta = 1, 2, 4$. So the energy density of one δ-potential is a finite quantity. Near the plane with the δ-potential, i.e. for $x_3 \to 0$, it diverges as $(x_3)^{-4}$, as expected from dimensional considerations.

Second, inserting (3.13) for $\overline{D^{\mathrm{c}}}(x,y)$ into (4.1) we get the contribution of two δ-potentials:

$$\langle 0|T_{00}^{2\delta}(x)|0\rangle = \mathrm{i}a\int \frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}k_0^2 \times \frac{(1-\mathrm{i}a/\Gamma)\,\mathrm{e}^{2\mathrm{i}\Gamma|x_3-d_1|} + (\mathrm{i}a/\Gamma)\,\mathrm{e}^{\mathrm{i}\Gamma(|x_3-d_1|+|x_3-d_2|+d)} + (d_1 \leftrightarrow d_2)}{(\Gamma - \mathrm{i}a)^2 + a^2\,\mathrm{e}^{2\mathrm{i}\Gamma d}}.$$

This is a finite expression for $x_3 \neq d_i$, where it diverges as for the contribution of one δ-potential. So we subtract the individual contributions from a single δ-potential at $x_3 = d_1$ and at $x_3 = d_2$:

$$\begin{aligned}\langle 0|T_{00}^{2\delta,\mathrm{sub}}(x)|0\rangle &= \langle 0|T_{00}^{2\delta}(x)|0\rangle - \langle 0|T_{00}^{1\delta}(x-d_1)|0\rangle - \langle 0|T_{00}^{1\delta}(x-d_2)|0\rangle \\ &= a^2\int \frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\frac{k_0^2\,\mathrm{e}^{\mathrm{i}\Gamma d}}{\Gamma(a+\mathrm{i}\Gamma)[(\Gamma-\mathrm{i}a)^2 + a^2\,\mathrm{e}^{2\mathrm{i}\Gamma d}]} \\ &\quad \times [a\,\mathrm{e}^{\mathrm{i}\Gamma(2|x_3-d_1|+d)} - (a+\mathrm{i}\Gamma)\,\mathrm{e}^{\mathrm{i}\Gamma(|x_3-d_1|+|x_3-d_2|)} + (d_1 \leftrightarrow d_2)]\end{aligned}$$

whereby the energy is shifted by an amount which is independent of the distance d between the planes with δ-potentials. For the distance-dependent part of the energy density per unit area

$$E_0 = \int_{-\infty}^{+\infty} \mathrm{d}x_3 \langle 0 | T_{00}^{2\delta,\mathrm{sub}}(x_3) | 0 \rangle \tag{4.2}$$

we get

$$E_0 = \int \frac{\mathrm{d}^3 k_\alpha}{(2\pi)^3} \frac{2a^2 k_0^2 [d - 1/(a + \mathrm{i}\Gamma)]}{\Gamma[(a+\mathrm{i}\Gamma)^2 - a^2 \,\mathrm{e}^{2\mathrm{i}\Gamma d}]} \mathrm{e}^{2\mathrm{i}\Gamma d}. \tag{4.3}$$

This formula represents the Casimir energy between two δ-potentials. In order to discuss its behaviour for different values of the parameter d we simplify (4.3). The analytic structure of the integrand contains an additional pole at $\Gamma = \mathrm{i}a$ which corresponds to the bound state of one individual δ-potential introduced by the subtraction procedure in (4.2). So the Wick rotation $k_0 \to \mathrm{i}k_4$ is possible and we get

with $\gamma = \sqrt{k_4^2 + k_1^2 + k_2^2 + m^2}$. The integration over the angles leads to

$$E_0 = -\frac{a^2}{3\pi^2} \int_0^\infty \mathrm{d}k \frac{k^4 [d + 1/(\gamma - a)]}{\gamma[(a-\gamma)^2 - a^2 \,\mathrm{e}^{-2\gamma d}]} \mathrm{e}^{-2\gamma d}$$

with $\gamma = \sqrt{k^2 + m^2}$. It is interesting to consider some limiting cases. For this reason we change the integration variable k to $y = \sqrt{k^2 + m^2} - m$ and obtain

$$E_0 = -\frac{a^2}{3\pi^2} \int_0^\infty \mathrm{d}y \frac{[y(y+2m)]^{3/2} [d + 1/(y + m - a)]}{(y + m - a)^2 - a^2 \,\mathrm{e}^{-2d(y+m)}} \mathrm{e}^{-2(y+m)d}. \tag{4.4}$$

Now, for $md \gg 1$, E_0 decreases exponentially, and to leading order we have

$$E_0 \underset{md \gg 1}{\sim} \frac{-a^2}{8(m-a)^3 d} \left(\frac{m}{\pi d}\right)^{3/2} \mathrm{e}^{-2md}. \tag{4.5}$$

The other limiting case is the massless one:

$$E_0^{(m=0)} = -\frac{a^2}{3\pi^2} \int_0^\infty \mathrm{d}\gamma \frac{\gamma^3 [d + 1/(\gamma - a)]}{(\gamma - a)^2 - a^2 \,\mathrm{e}^{-2\gamma d}} \mathrm{e}^{-2\gamma d}.$$

In the limit $ad \to -\infty$ we get the known result corresponding to the Dirichlet boundary conditions,

$$E_0^{(m=0)} \underset{ad \to -\infty}{\sim} \frac{-\pi^2}{720 d^3} \tag{4.6}$$

and for weak δ-potentials we get

$$E_0^{(m=0)} \underset{ad \to 0}{\sim} \frac{-a^2}{2\pi^2 d}. \tag{4.7}$$

These results show the expected behaviour, especially the exponential suppression in the massive case. In all these cases the Casimir force is attractive.

In the case of the spinor field the calculations are similar; however, they are technically more complicated. The vacuum expectation value of T_{00} (using (2.5) and (2.6)) is

$$\langle 0 | T_{00}(x) | 0 \rangle = -\frac{1}{2} \left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial y^0} \right) \mathrm{Tr}\, \gamma^0 \, {}^{s}S^{c}(x, y) \big|_{x=y}.$$

Using (3.20), (3.21) and omitting the distance-independent contribution containing $S^c(x, y)$ we get

$$\langle 0|T_{00}(x)|0\rangle = \int \frac{d^3p_\alpha}{(2\pi)^3}\frac{p_0}{2\Gamma}\sum_{i,j=1}^{n} \mathrm{Tr}\,\gamma^0\{[\check{p}+m-\gamma^3\Gamma\varepsilon(x_3-d_i)]K_{ij}$$
$$\times[\check{p}+m-\gamma^3\Gamma\varepsilon(d_j-x_3)]\}\,e^{i\Gamma(|x_3-d_i|+|x_3-d_j|)}$$

which is finite for $x_3 \neq d_i$ $(i=1,2)$. As in the scalar case we subtract the contributions from individual δ-potentials at $x_3=d_i$. By means of (3.27) this results just in the omission of the first term in (3.22) for K_{ij}. Now, we consider the contributions to the vacuum energy resulting from three different regions of the x_3 axis separately. Defining

$$E^2 = \int_{-d/2}^{d/2} dx_3\langle 0|T_{00}^{\mathrm{sub}}(x)|0\rangle$$

$$E^3 = \int_{d/2}^{\infty} dx_3\langle 0|T_{00}^{\mathrm{sub}}(x)|0\rangle$$

$$E^1 = \int_{-\infty}^{-d/2} dx_3\langle 0|T_{00}^{\mathrm{sub}}(x)|0\rangle$$

we have

$$E = E^1+E^2+E^3.$$

In the region between the δ-potentials, i.e. for $-d/2<x_3<d/2$ we get

$$\langle 0|T_{00}^{\mathrm{sub}}(x)|0\rangle = \int \frac{d^3p_\alpha}{(2\pi)^3}\frac{p_0\alpha^2\delta^2 m}{\Gamma N}\mathrm{Tr}\,\gamma^0$$
$$\times(\alpha\delta P_-A^{-1}P_+A^{-1}P_-A^{-1}P_+\,e^{2i\Gamma x_3}+\alpha\delta P_+A^{-1}P_-A^{-1}P_+A^{-1}P_-\,e^{-2i\Gamma x_3}$$
$$-P_+A^{-1}P_-A^{-1}P_+-P_-A^{-1}P_+A^{-1}P_-).$$

Using (3.25) as well as the traces

$$\mathrm{Tr}\,\gamma^0P_+\gamma^0P_- = 2(p_1^2+p_2^2+m^2)/m^2 \qquad \mathrm{Tr}\,\gamma^0P_\pm = 2p_0/m$$

we get for E^2 the contribution

$$E^2 = -\int \frac{d^3p_\alpha}{(2\pi)^3}\frac{2p_0\alpha^2\delta^2(p_1^2+p_2^2+m^2)}{\lambda^2m^2\Gamma N}\left(p_0d+i\frac{\alpha(p_1^2+p_2^2+m^2)(\delta^2-1)}{m\lambda\Gamma}\right). \tag{4.8}$$

For the region $d/2<x_3$ we obtain

$$\langle 0|T_{00}^{\mathrm{sub}}(x)|0\rangle = \int \frac{d^3p_\alpha}{(2\pi)^3}\frac{p_0\alpha^2\delta m}{\Gamma N}\mathrm{Tr}\,\gamma^0$$
$$\times(\alpha\delta^2P_-A^{-1}P_+A^{-1}P_-A^{-1}P_++\alpha P_-A^{-1}P_-A^{-1}P_+A^{-1}P_+$$
$$-P_-A^{-1}P_+A^{-1}P_--P_-A^{-1}P_-A^{-1}P_-)\,e^{2i\Gamma x_3}.$$

With the help of the auxiliary formulae

$$P_\pm A^{-1}P_\pm = \frac{\mu}{\lambda m}P_\pm \qquad \text{with } \mu = p_0 - i\gamma\tan(\vartheta/2)$$

we obtain for E^3

$$E^3 = \mathrm{i} \int \frac{\mathrm{d}^3 p_\alpha}{(2\pi)^3} \frac{p_0 \alpha^2 \delta^2}{\lambda^2 m^2 \Gamma^2 N} \left[\mu^2 p_0 - (p_1^2 + p_2^2 + m^2) \left(\frac{\alpha \delta^2}{\lambda m} (p_1^2 + p_2^2 + m^2) + \frac{\alpha \mu^2}{\lambda m} - p_0 \right) \right]. \tag{4.9}$$

Clearly, from the region $x_3 < -d/2$ we get the same contribution, i.e. $E^1 = E^3$.

Finally, to make these expressions more transparent, we consider two limiting cases.

First, we assume $md \gg 1$. In (4.8) for E^2 we perform the Wick rotation and consider the leading contribution only. The result is

$$E^2 \underset{ad \gg 1}{\sim} \frac{1}{4} \tan \vartheta \left(\frac{m}{\pi d} \right)^{3/2} \mathrm{e}^{-2md} \tag{4.10}$$

which shows the expected proportionality to $\exp(-2md)$. It can be shown that the contribution from E^1 and E^3 are by one power $1/md$ smaller in this limiting case. Note that the Casimir effect is a long-distance effect by comparing the distance d with the Compton wavelength of the corresponding massive particle. Therefore, exponentially vanishing contributions are suppressed in comparison with those of massless particles.

The other limiting case is the massless one. Here all contributions (i.e. from E^2 and from E^1 and E^3) are of equal order and the energy for small ϑ is

$$E^{(m=0)} \underset{\vartheta \sim 0}{\sim} \frac{\vartheta^2}{15 \pi^2 d^3}. \tag{4.11}$$

Note that the Casimir force for the spinor case is repulsive. This is in contrast to the scalar case. From the viewpoint of supersymmetry [11] one may believe that spinor contributions have an opposite sign in comparison to the scalar ones. In [7] other boundary conditions are used, leading to an attractive force for the spinor case too.

5. The non-relativistic approximation

It is interesting to compare the calculation of the Casimir force corresponding to massive fields in relativistic and non-relativistic quantum field theory. A heuristic argument could be as follows. For the Casimir effect only low frequencies are essential. The reason is that the high frequencies cancel themselves when the free space contribution is subtracted. On the other hand, the low-energy behaviour for massive fields is considered to be well described by a non-relativistic approximation. We show that this is not the case. From the foregoing sections and from [7] we know that for large m any Casimir force decreases exponentially as $\exp(-2md)$. This result is not reproduced in non-relativistic quantum field theory.

We consider the Schrödinger equation

$$(\mathrm{i}\partial_t - \hat{H}) \psi(x) = 0 \tag{5.1}$$

with the Hamiltonian

$$\hat{H} = -\frac{\nabla^2}{2m} + eA_0(x)$$

where the potential is given by (1.1). The second quantized field operators are

$$\psi(x)=\sum_\alpha \mathrm{e}^{+\mathrm{i}\varepsilon_\alpha t}\Phi^*_\alpha(x)a^-_\alpha$$

$$\psi^*(x)=\sum_\alpha \mathrm{e}^{-\mathrm{i}\varepsilon_\alpha t}\Phi_\alpha(x)a^+_\alpha$$

with

$$[a^-_\alpha, a^+_\alpha]=\delta_{\alpha\alpha'}$$

and $\Phi_\alpha(x)$ are the eigenfunctions of the Hamiltonian

$$\hat{H}\Phi_\alpha(x)=\varepsilon_\alpha\Phi_\alpha(x). \tag{5.2}$$

Taking into account a symmetric operator ordering in the Hamiltonian we get for the ground state energy

$$E_0^{(\mathrm{nr})}=\tfrac{1}{2}\sum_\alpha \varepsilon_\alpha. \tag{5.3}$$

This can be represented with the help of the retarded propagator $\Delta(x, y)$ as

$$E_0^{(\mathrm{nr})}=\mathrm{i}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\mathrm{d}x_3\,\Delta(x,y)\bigg|_{\substack{x_0-y_0\to+0\\ x=y}}. \tag{5.4}$$

Because of the retardation property of the propagator (5.4) we use $t-t'>0$ as a regularization.

Consider first the free field case. Here the propagator in the mixed representation is

$$\Delta(x,y)=\int\frac{\mathrm{d}^3k_\alpha}{(2\pi)^3}\frac{m}{\Gamma}\,\mathrm{e}^{\mathrm{i}k_\alpha(x^\alpha-y^\alpha)+\mathrm{i}\Gamma|x_3-y_3|} \tag{5.5}$$

where Γ is given by

$$\Gamma=\sqrt{2mk_0-k_1^2-k_2^2+\mathrm{i}\varepsilon}. \tag{5.6}$$

This is the only change introduced by non-relativistic field theory in comparison with the corresponding relations (3.10), (3.11) for relativistic field theory. With this change all the further formulae of section 3 for δ-potentials are valid in this case too.

Now, the calculation of the distance-dependent part of the vacuum energy E_0 per unit area is performed in the same way as in section 4 and we arrive at

$$E_0^{(\mathrm{nr})}=-2a^2m\int\frac{\mathrm{d}_E^2k}{(2\pi)^3}\frac{k_0[d-1/(a+\mathrm{i}\Gamma)]}{\Gamma[(a-\mathrm{i}\Gamma)^2-a^2\,\mathrm{e}^{2\mathrm{i}\Gamma}]}\,\mathrm{e}^{2\mathrm{i}\Gamma d-\mathrm{i}k_0(t-t')}. \tag{5.7}$$

Here the difference between the definitions of the energy, i.e. between (4.1) and (5.4) as well as (5.6) are taken into account. Further, we used $t>t'$ as a regularization. Now the analytic structure of the integrand on the RHS of (5.7) differs from that in the corresponding expressions in section 4 because Γ contains $2mk_0$ instead of k_0^2. Therefore, the cut and the poles on the left side of the k_0 plane are absent. So, if the integration path could be closed in the upper half plane the result would be zero. Now we argue that the regularization $t>t'$ can be removed when the result is finite. This is possible. For example, we can give the distance d a positive imaginary part $d\to d+\mathrm{i}\varepsilon$. Then $\exp(2\mathrm{i}\Gamma d)$ makes the integral convergent and it is possible to set $t-t'=0$. Note that this is closely connected with the fact that the Casimir force is finite. So the non-relativistic energy (5.7) vanishes: $E_0^{(\mathrm{nr})}=0$.

To understand this result better we consider the more simple case of two plates, represented by Dirichlet boundary conditions. We perform a direct summation of the zero point energies with the help of the ζ-function method. Here, $\Sigma_{(\alpha)}$ represents integration over the moments k_1 and k_2 parallel to the plates and summation over the perpendicular frequencies $k_3 = \pi n/d$ $(n = 1, 2, \ldots)$. The energy is $\varepsilon_{(\alpha)} = [k_1^2 + k_2^2 + (\pi n/d)^2]/2m$ and we get

$$E_0^{(\mathrm{nr})} = \frac{1}{2} \int \frac{\mathrm{d}k_1\, \mathrm{d}k_2}{(2\pi)^2} \sum_{n=1}^{\infty} \left(\frac{k_1^2 + k_2^2 + (\pi n/d)^2}{2m} \right)^{\sigma} \tag{5.8}$$

where σ is a regularization parameter with $\sigma \to 1$ finally. Equation (5.8) can be easily calculated:

$$\begin{aligned} E_0^{(\mathrm{nr})} &= \frac{-1}{8\pi(2m)^{\sigma}(1+\sigma)} \sum_{n=1}^{\infty} \left(\frac{\pi n}{d} \right)^{2\sigma+2} \\ &= \frac{-1}{4\pi(2m)^{\sigma}(1+\sigma)} \left(\frac{\pi}{d} \right)^{2+2\sigma} \zeta(-2-2\sigma) \end{aligned} \tag{5.9}$$

where ζ is the ζ-function. In the limit $\sigma \to 1$ we clearly get $E_0^{(\mathrm{nr})} = 0$ because of the zeros of the ζ-function.

These examples allow the conclusion that the Casimir effect of a massive field has an essential relativistic character and cannot be calculated within a non-relativistic theory. In our case the non-relativistic ground state energy of the Schrödinger field theory does not depend on the parameters of the system [15]. In a pure mathematical sense this means that the large mass limit (i.e. $mc^2 \to \infty$) in relativistic theory has to be performed after calculating the ground state energy. An interchange of this operation is not allowed, as the expansion $k_0 = m + k_{(\alpha)}^2/2m + \ldots$ cannot be used. So the heuristic argument that high frequencies are not essential for the Casimir force fails.

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